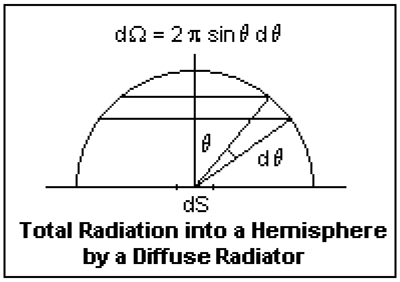
Derivation of the relationship between illuminance E and luminance L for a Lambertian reflective surface, $L = E\rho / \pi$.

For this illustration to keep it as clear as I can, let's assume for now the reflectance $\rho = 1$.



Total Radiation into a Hemisphere by a Diffuse Radiator

$$d\Phi_h = \int_0^{2\pi} d\varphi \int_0^{\pi/2} L\, dA \cos\theta \sin\theta\, d\theta = \pi L\, dA$$

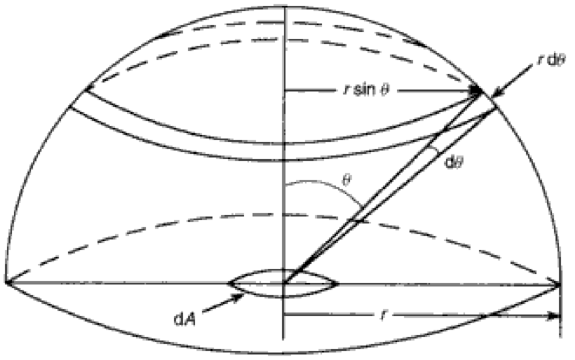


Figure 5.3

Suppose you are looking at a small illuminated diffuse reflector of area dS from a certain angle, and receive a flux of dF lumens from it. The projected area normal to your line of vision is dScosθ. If you look at it from a different direction, the projected area may change, but the area will look equally bright, which means the same flux per unit projected area.

This flux is emitted per solid angle dΩ and so we can find the flux per unit solid angle as well (which is the intensity in candela, by definition), which will allow us to integrate the flux over any surface.

Then, for Lambertian surfaces only, the constant $L = dF / d\Omega\, dS\cos\theta$.

So $dF = L\, dS\cos\theta\, d\Omega$.

To find the total light emitted by dS, we integrate over $d\Omega=2\pi \sin\theta\, d\theta$ from $\theta = 0$ to $\pi/2$ (from the horizontal to the vertex, which is $\pi/2$, and over the entire hemisphere) – see figure above for the 3-d spatial representation (I found two useful pictures which represent the same thing).

The result is $dF = \pi L\, dS$,

So the constant $L = (1/\pi)dF/dA = E/\pi$, or the total flux emitted divided by π.

If a diffuse surface receives E lumens/m^2, then E/π is its surface brightness in cd/m2, and the light emitted at an angle θ into solid angle $d\Omega$ is $(E/\pi)\, dS\cos\theta\, d\Omega$. The total light emitted from dS is then E.

Now if the reflectance of the surface is not 1, then just use the appropriate reflectance value ρ in the equation:

$L = E\, \rho / \pi$

The ratio of the illuminance E to the Luminace L of a lambertian surface is a factor of π and NOT 2π : this is somewhat counterintuitive since there are 2π steradians in a hemisphere. But remember this is for Lambertian surface and we are NOT talking about a point source, such as a light bulb. The $\cos\theta$ factor is responsible for this reduction.

You may ask why are lambertian surfaces important. Here are some examples of real situations: surface painted with a good “matte” or “flat” white paint, if uniformly illuminated like by the sun, will appear equally bright from whatever direction you view. The Sun is almost a perfect Lambertian radiator, and as a result the brightness of the Sun is almost the same everywhere on an image of the solar disk even though we see it full-on only in the center of the disk. Many naturally occurring surfaces exhibit Lambertian characteristics up to θ=40 deg. In satellite observations, snow and desert is Lambertian up to 50-60 deg. Most surfaces depart significant from Lambertian above 60 deg, except for the White Sands desert in New Mexico, which is Lambertian for nearly all angles….. interesting!

Now…. if you are adverse to all that integral calculus above, there is another interest derivation that is geometrical (Photometry and Radiometry – a tour guide for computer graphics enthusiasts, by Ian Ashdown, President byHeart Consultants Ltd., October 2002). Need some abstract thinking though – about that imaginary sphere-, when applying the principles.

6. Lambertian Surfaces

A Lambertian surface is a surface that has a constant radiance or luminance that is independent of the viewing direction. In accordance with the definition of radiance (luminance), the radiant (luminous) flux may be emitted, transmitted, and/or reflected by the surface.

A Lambertian surface is also referred to as an *ideal diffuse* emitter or reflector. In practice there are no true Lambertian surfaces. Most matte surfaces approximate an ideal diffuse reflector but typically exhibit semispecular reflection characteristics at oblique viewing angles. Nevertheless, the Lambertian surface concept is useful in computer graphics.

Lambertian surfaces are unique in that they reflect incident flux in a completely diffuse manner (Fig. 7). It does not matter what the angle of incidence θ of an incoming geometrical ray is -- the distribution of light leaving the surface remains unchanged.

We can imagine a differential area dA of a Lambertian surface. Being infinitesimally small, it is equivalent to a point source, and so the flux leaving the surface can be modeled as geometrical rays. The intensity I_θ of each ray leaving the surface at an angle θ from the surface normal is given by *Lambert's cosine law*:

$$I_\theta = I_n \cos\theta \tag{15}$$

where I_n is the intensity of the ray leaving in a direction perpendicular to the surface.

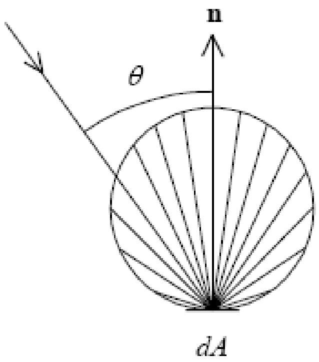


Figure 7 Reflection from a Lambertian surface

The derivation of Equation 15 becomes clear when we remember that we are viewing dA from an angle θ. For a differential area dA with a constant radiance or luminance, its intensity must vary in accordance with its projected area, which is $dA\cos\theta$. This gives us:

$$L = dI/(dA\cos\theta) = dI_n/dA \tag{16}$$

for any Lambertian surface.

There is a very simple relation between radiant (luminous) exitance and radiance (luminance) for flux leaving a Lambertian surface:

$$M = \pi L \tag{17}$$

where the factor of π is a source of endless confusion to students of radiometry and photometry. Fortunately, there is an intuitive explanation. Suppose we place a differential Lambertian emitter dA on the inside surface of an imaginary sphere S (Fig. 8). The inverse square law (Eqn. 12) provides the irradiance E at any point P on the inside surface of the sphere. However, $d = D\cos\theta$, where D is the diameter of the sphere. Thus:

$$E = I_\theta \cos\theta/(D\cos\theta)^2 = I_\theta/D^2 \cos\theta \tag{18}$$

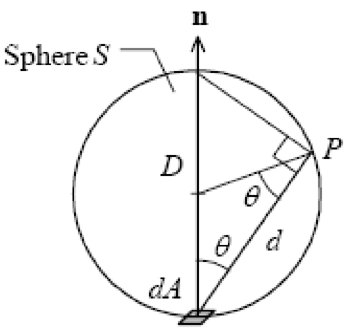


Figure 8 A Lambertian emitter illuminating the interior of a sphere

and from Lambert's cosine law (Eqn. 15), we have:

$$E = I_\theta \cos\theta / D^2 \cos\theta = I_n / D^2 \tag{19}$$

which simply says that the irradiance (radiant flux density) of any point P on the inside surface of S is a constant.

This is interesting. From the definition of irradiance (Eqn. 4), we know that $\Phi = EA$ for constant flux density across a finite surface area A. Since the area A of the surface of a sphere with radius r is given by:

$$A = 4\pi r^2 = \pi D^2 \tag{20}$$

we have:

$$\Phi = EA = \pi I_n D^2 / D^2 = \pi I_n \tag{21}$$

Given the definition of radiant exitance (Eqn. 5) and radiance for a Lambertian surface (Eqn. 16), we have:

$$M = d\Phi / dA = \pi dI_n / dA = \pi L \tag{22}$$

This explains, clearly and without resorting to integral calculus, where the factor of π comes from.

Many people have found reflection and luminance to be a difficult concept to grasp. Yet it is so critical because it directly relates to what we see around us, including design issues of contrast, glare, color and architectural form rendition, etc.